



Non-autonomous Dynamics in Complex Systems: Theory and Applications to Critical Transitions

Dresden, 16-20 October 2023



The Pullback Attractors of an Excitable Low-Order Ocean Model with Periodic, Aperiodic and Monotonically Drifting Forcing

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The problem

The model

- Periodic forcing
- Aperiodic forcing
- Monotonically drifting forcing

Conclusions

The problem

This talk is about a **nonautonomous dynamical system** and its **pullback attractors** corresponding to different forcing agents

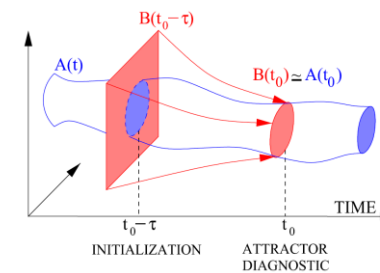
The stress will be put on the ability of such time-dependent objects to **reveal fundamental and often unexpected dynamical features**

The model used here is low-dimensional and derives from a relevant oceanographical problem. However, this should be seen as a **prototype of an excitable nonlinear dynamical system with time-dependent forcing** that may evidence basic features of climate change in the presence of natural variability

The problem

The **correct description of a changing climate** subjected to both natural and anthropogenic forcing requires the knowledge of the time-dependent probability distribution associated with the climate's ***pullback/snapshot attractor (PBA)*** (e.g., Romeiras et al. 1990; Ghil et al 2008; Chekroun et al., 2011; Bódai et al. 2011; Drótos et al. 2015, 2016; Ghil and Lucarini 2020; Tél et al. 2020, Pierini and Ghil 2021)

The PBA is a mathematical tool which provides the **extension to a nonautonomous dissipative dynamical system** (for which the external forcing and/or some parameter depend on time) **of the classical concept of attractor of the corresponding autonomous system**



From
Sévellec &
Fedorov
(2015, EPSL)

The resulting description can be **considerably different from the classical one based on the 30-year temporal average of a single long time series** of climatological data, whether observed or simulated numerically: this is basically due to the nonergodicity of aperiodically forced systems (e.g., Drótos et al. 2016)

The PBA is estimated numerically by performing **an ensemble of many forward time integrations differing only by their respective initializations**. The integrations go well beyond the system's predictability time

The analysis of the system's PBA **reveals some fundamental and sometimes unexpected dynamical features**

The problem

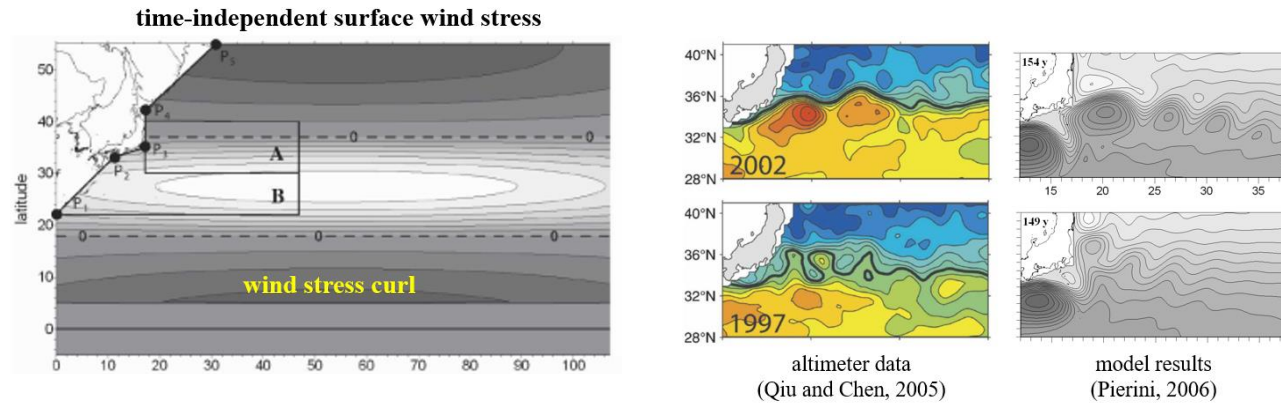
The model

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The model

The **Kuroshio Extension intrinsic low-frequency variability** was previously studied by means of a system of **3 PDEs** (Pierini 2006, 2014a, Pierini, Dijkstra & Riccio, 2009, etc.)



A low-order model (**4 ODEs**) was then developed (Pierini 2011) to obtain an agile mathematical tool with which ensemble simulations could easily be carried out

In this presentation **the PBAs of such excitable low-order quasigeostrophic ocean model** are discussed

The model

Pierini (2011,
J. Phys. Oceanogr.)

The model is based on the evolution equation of potential vorticity in the QG reduced-gravity approximation,

$$\frac{\partial}{\partial \bar{t}} \left(\bar{\nabla}^2 \bar{\psi} - \frac{\bar{\psi}}{L_R^2} \right) + \bar{J}(\bar{\psi}, \bar{\nabla}^2 \bar{\psi}) + \beta \bar{\psi}_{\bar{x}} = -r \bar{\nabla}^2 \bar{\psi} + \frac{\text{curl}_{\bar{z}} \bar{\tau}}{\rho H}$$

that can be adimensionalized as follows:

$$\frac{\partial}{\partial t} (\nabla_{\lambda}^2 \psi - F\psi) + \gamma J(\psi, \nabla_{\lambda}^2 \psi) + \psi_x = -R \nabla_{\lambda}^2 \psi - T \tau_y$$

The truncated spectral model is obtained by expanding the streamfunction ψ as:

$$\psi(\mathbf{x}, t) = \sum_{i=1}^4 \Psi_i(t) |i\rangle$$

$$\begin{aligned} |1\rangle &= e^{-\alpha x} \sin x \sin y & |3\rangle &= e^{-\alpha x} \sin 2x \sin y \\ |2\rangle &= e^{-\alpha x} \sin x \sin 2y & |4\rangle &= e^{-\alpha x} \sin 2x \sin 2y \end{aligned}$$

This choice follows that of Jiang et al. (1995), who adopted $|i\rangle$, $i=1,2$ in their 2D model. In the Hilbert space endowed with the inner product

$$\langle f | g \rangle = \frac{4}{\pi^2} \int_0^{\pi} \int_0^{\pi} e^{2\alpha x} f g \, dx dy$$

the basis is orthonormal: $\langle i | j \rangle = \delta_{ij}$.

The system reduces to a set of four coupled nonlinear ODEs:

$$\begin{cases} \dot{\Psi}_1 + p\Psi_1 + q\Psi_3 + N_1 = W_1 \\ \dot{\Psi}_2 + u\Psi_2 + v\Psi_4 + N_2 = W_2 \\ \dot{\Psi}_3 + m\Psi_3 + o\Psi_1 + N_3 = W_3 \\ \dot{\Psi}_4 + s\Psi_4 + t\Psi_2 + N_4 = W_4 \end{cases}$$

The nonlinear terms N_i are given by

$$N_i = \sum_{j,k=1}^4 \Psi_j J_{ijk} \Psi_k$$

so that the system can be written in compact form as follows:

$$\frac{d\Psi}{dt} + \mathbf{J}\Psi + \mathbf{L}\Psi = \mathbf{W} \quad \mathbf{W} = \mathbf{G}(t)\mathbf{w}$$

The rank-3 tensor \mathbf{J} is given by:

$$\begin{aligned} J_{1jk} &= \frac{c_1 Z_{1jk} - h_1 Z_{3jk}}{h_1^2 + a_1 c_1}, & J_{2jk} &= \frac{d_1 Z_{2jk} - h_1 Z_{4jk}}{h_1^2 + b_1 d_1} \\ J_{3jk} &= \frac{h_1 Z_{1jk} + a_1 Z_{3jk}}{h_1^2 + a_1 c_1}, & J_{4jk} &= \frac{h_1 Z_{2jk} + b_1 Z_{4jk}}{h_1^2 + b_1 d_1} \end{aligned}$$

where

$$Z_{ijk} \equiv \langle i | Q_{jk} \rangle$$

and where \mathbf{Q} is the symmetric tensor:

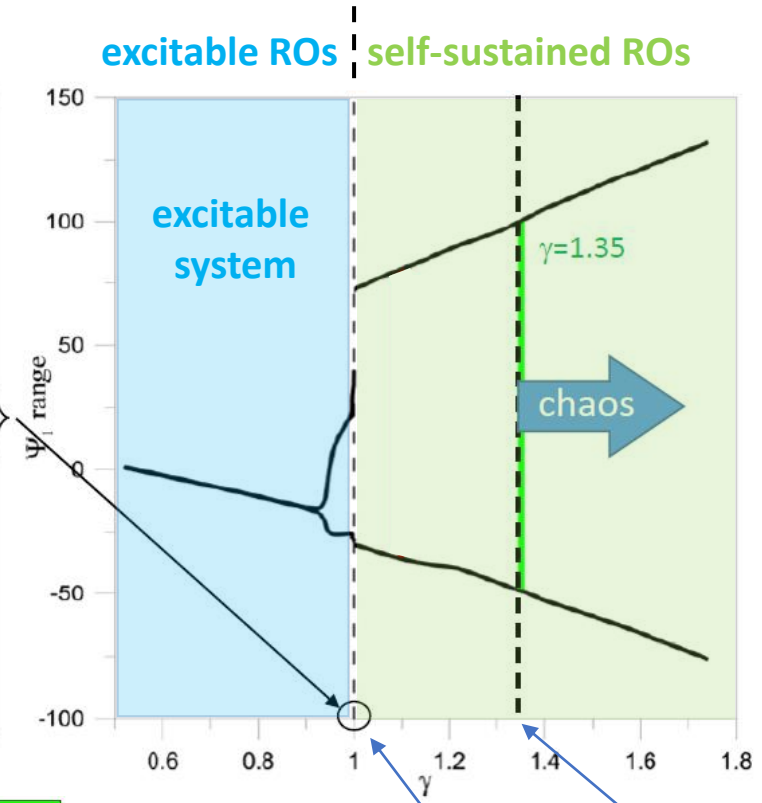
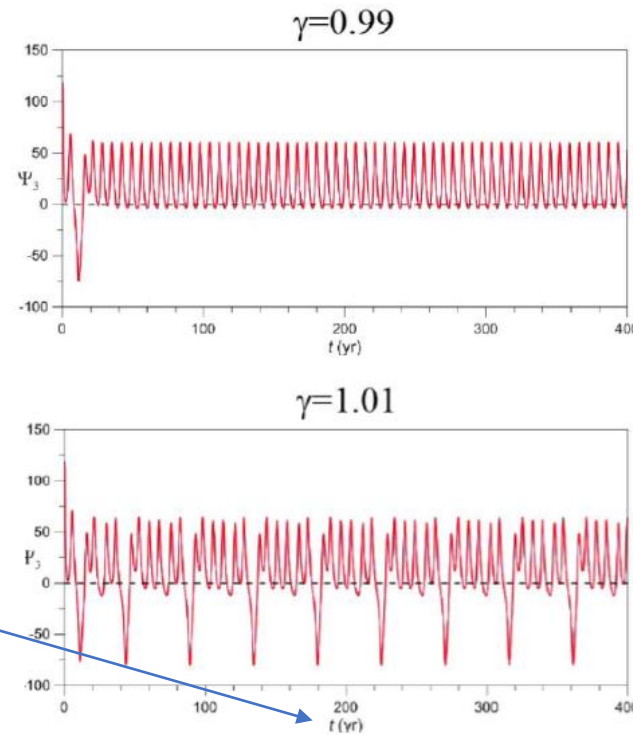
$$\begin{aligned} |Q_{jk}\rangle &= \frac{\gamma}{2} [|j\rangle_x (|k\rangle_{xxy} + \lambda |k\rangle_{yyy}) + |k\rangle_x (|j\rangle_{xxy} + \lambda |j\rangle_{yyy}) \\ &\quad - |j\rangle_y (|k\rangle_{xxy} + \lambda |k\rangle_{yyy}) - |k\rangle_y (|j\rangle_{xxy} + \lambda |j\rangle_{yyy})] \quad \dots \end{aligned}$$

The model

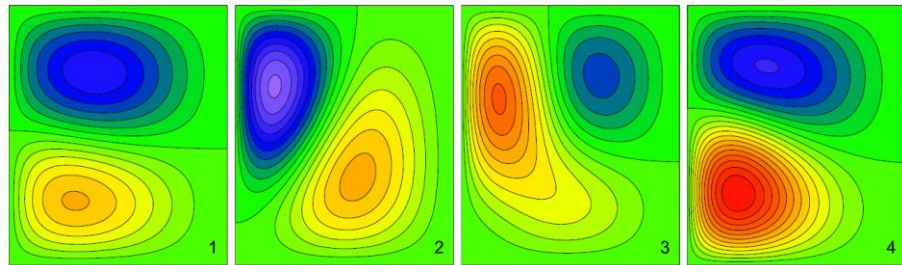
Pierini (2011,
J. Phys. Oceanogr.)

autonomous
system's behavior
($G = \gamma$)

dimensional t is used in the
graphs to keep memory of the
original physical problem



$\psi(\mathbf{x}, t)$:



transition to
large-amplitude
relaxation oscillations (ROs)

transition to chaos

The problem

The model

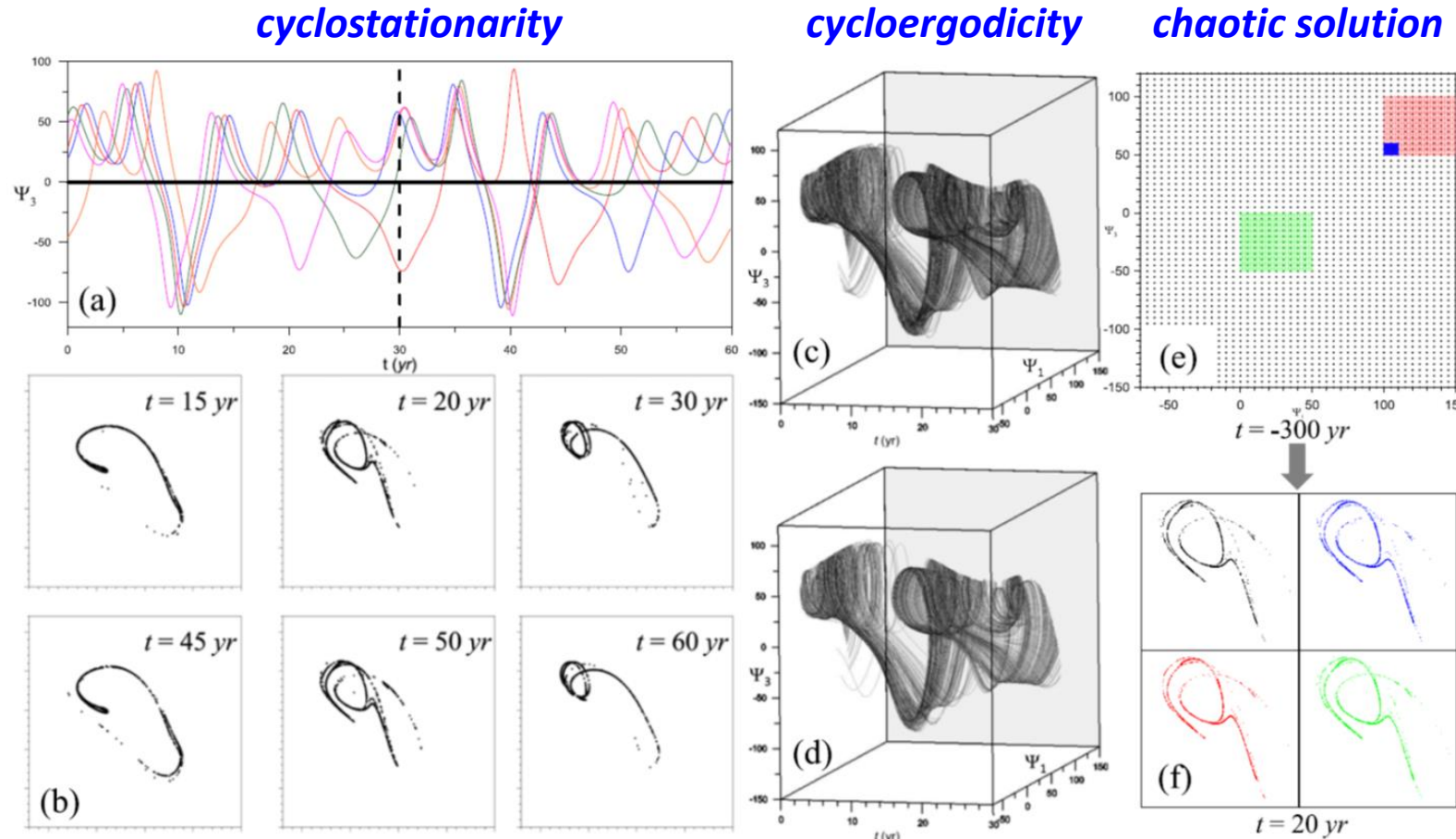
- **Periodic forcing**
- Aperiodic forcing
- Monotonically drifting forcing

Conclusions

$$G(t) = \gamma[1 + \varepsilon \sin(\omega t)]$$

Periodic forcing

Pierini (2014b, *J. Phys. Oceanogr.*)



$\gamma=1.1,$
 $T_p=30 \text{ yr},$
 $\varepsilon=0.2$

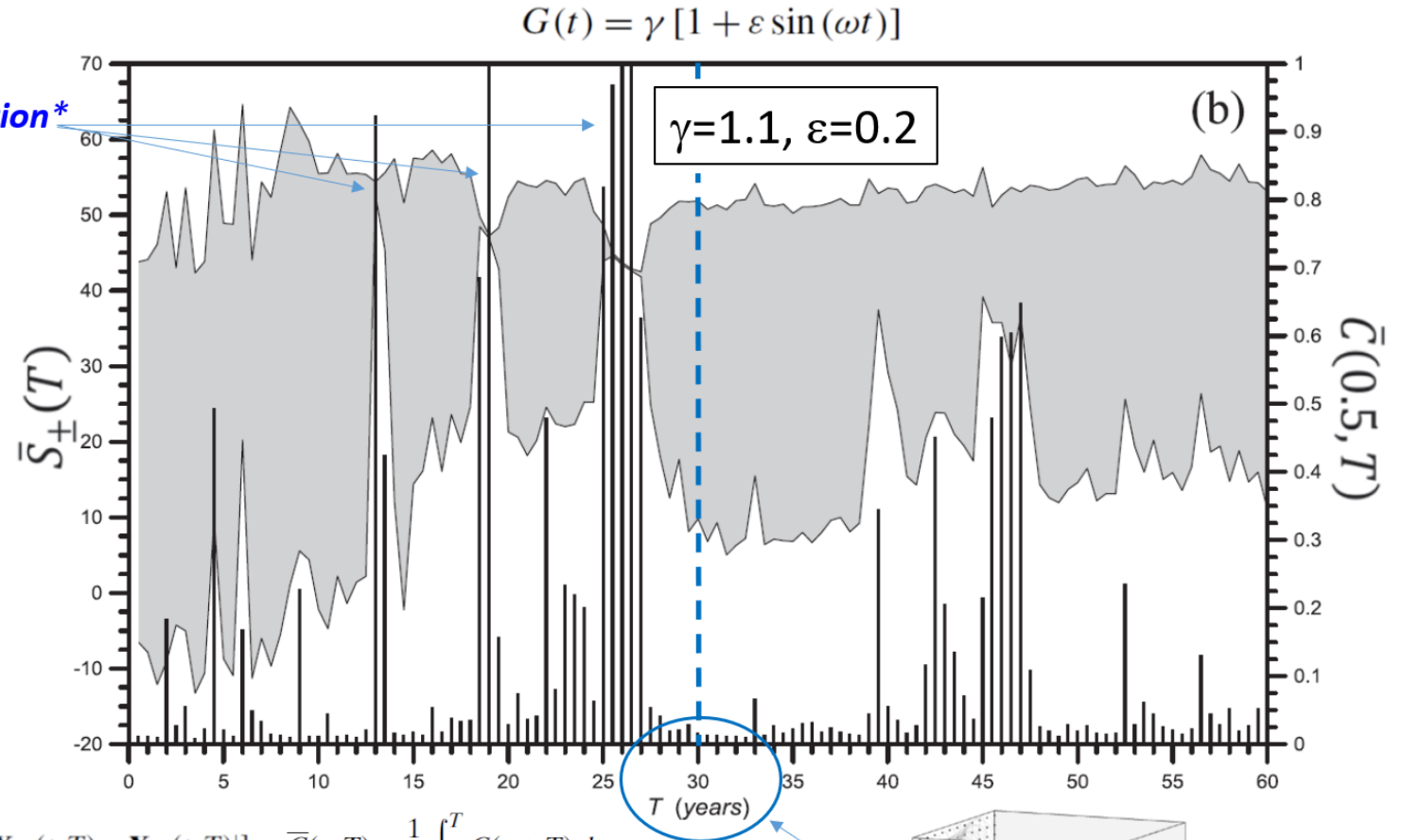
Note that the autonomous system with $\gamma=1.1$ is **nonchaotic**

FIG. 3. (a) Projection of some of the aperiodic trajectories of Fig. 2 onto the $t - \Psi_3$ plane. (b) Intersection of 2475 trajectories of the PBA of Fig. 2 with the $\Psi_1 - \Psi_3$ plane at the times indicated in the panels [the axes correspond to those of panels (c),(d)]; the patterns separated by a 30-yr cycle are identical (periodic PBA). (c) One cycle of the PBA of Fig. 2 obtained with 1800 trajectories. (d) Superposition of 1800 successive 30-yr intervals of a single trajectory (with the parameters of Fig. 2), whose initial times are all shifted back to $t = 0$. (f) Intersection of the PBA of Fig. 2 with the $\Psi_1 - \Psi_3$ plane at $t = 20 \text{ yr}$ for the four sets of ICs shown in panel (e).

Periodic forcing

Note the emergence of **generalised synchronization*** for specific ranges of the forcing period

*e.g., see Crucifix (2013) for examples relevant to climate dynamics



$$C(r, t, T) = \frac{1}{N^2} \sum_{k_1, k_2=1}^N H[r - |\mathbf{X}_{k_1}(t, T) - \mathbf{X}_{k_2}(t, T)|], \quad \bar{C}(r, T) = \frac{1}{T} \int_0^T C(r, t, T) dt$$

$(1/n) \leq \bar{C} \leq 1$, where n is the number of clusters defined by the distance r .
For $\bar{C} = 1$ one has only one cluster and, therefore, **TIID**.

TIID: *total independence from the initial data*

Periodic forcing

Pierini (2014b, *J. Phys. Oceanogr.*)

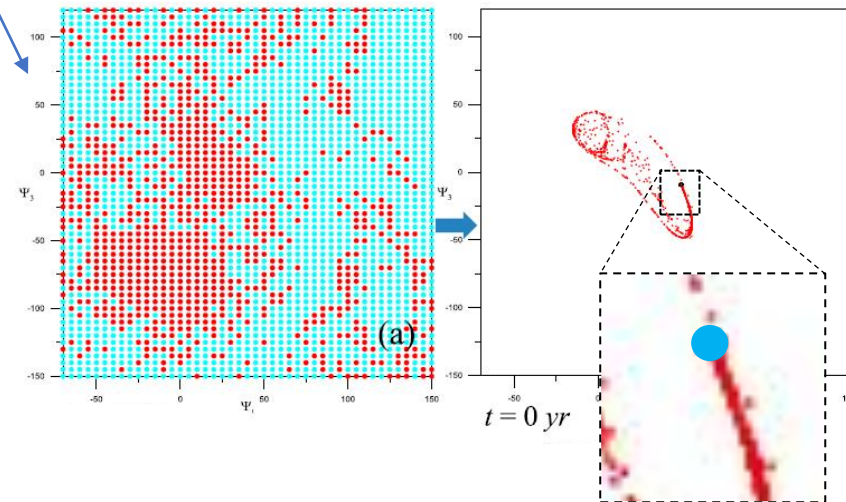
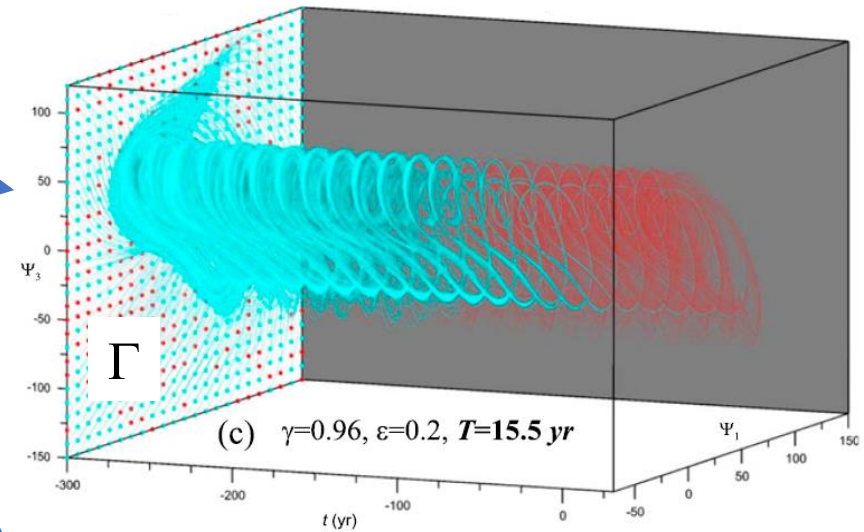
completely different evolutions
emerge from two basins of attraction,
as evidence in the plane set Γ

To characterize this behaviour, Pierini, Ghil
and Chekroun (2016) introduced a parameter σ :

$$\sigma(X, Y) = \frac{1}{T_*} \int_0^{T_*} \delta_n(t) dt$$
$$\delta_n(t) = \delta(t) / \delta(0)$$
$$(X, Y) \equiv (\Psi_1(0), \Psi_3(0)) \in \Gamma$$

σ : mean
normalized
distance

This turns out to be a valuable tool for any type
of forcing (see later)



Periodic forcing

Onset of chaos due to periodic forcing

Ghil (2019, *Earth Space Sci.*):

Classical dynamical systems theory deals with closed systems, which are governed by models in which there is no explicit time dependence whatsoever. These models are called autonomous, and the time independence property plays a key role in the theory. For instance, given a system of two ordinary differential equations (ODEs) in the plane, uniqueness of solutions essentially prevents self-intersection of the orbits, which can only tend to fixed points or limit cycles. It takes either periodic forcing, as is the case for the Duffing or Van der Pol oscillator, or three autonomous ODEs in Euclidean 3-D space, as for the Lorenz (1963a) model, to get deterministically irregular, chaotic behavior.

Transition to chaos has been considered very extensively in this case

Transition to chaos due to periodic forcing is less studied

$$G(t) = \gamma[1 + \varepsilon \sin(\omega t)]$$

Periodic forcing

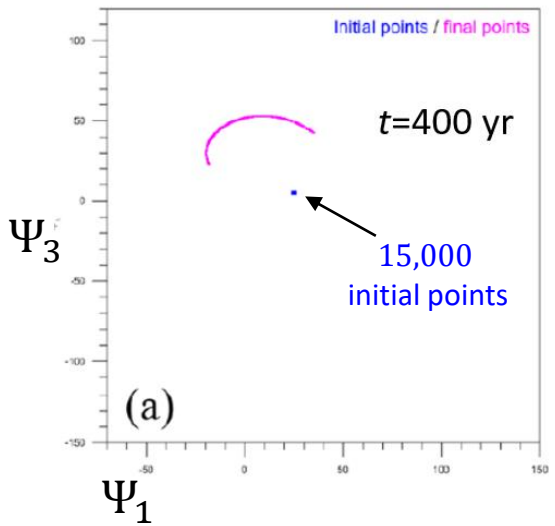
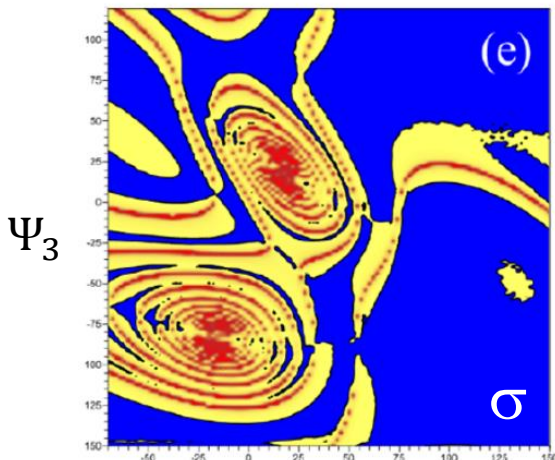
Pierini, Chekroun & Ghil (2018, *Nonlin. Processes Geophys.*)

autonomous *non-chaotic* case

$\varepsilon=0$

$\gamma=1.1, T_p=30 \text{ yr}$,

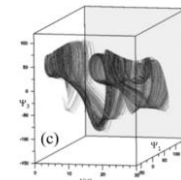
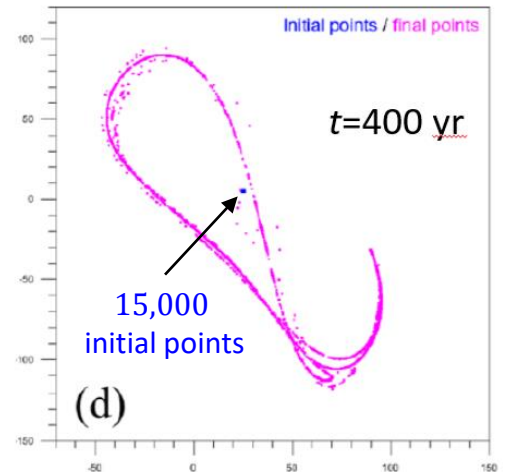
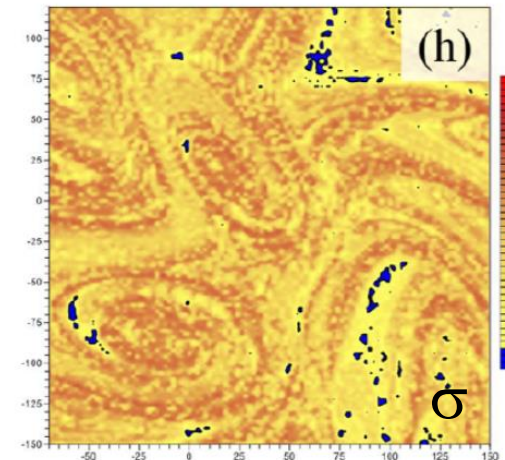
this is the **periodically forced *chaotic*** case shown in a previous slide



$\sigma > 1$ is a necessary, but **not** sufficient condition for chaos

transition to chaos

$\varepsilon=0.2$



$$G(t) = \gamma[1 + \varepsilon \sin(\omega t)]$$

Periodic forcing

Pierini, Chekroun & Ghil (2018, *Nonlin. Processes Geophys.*)

autonomous *non-chaotic* case

$\varepsilon=0$ ●

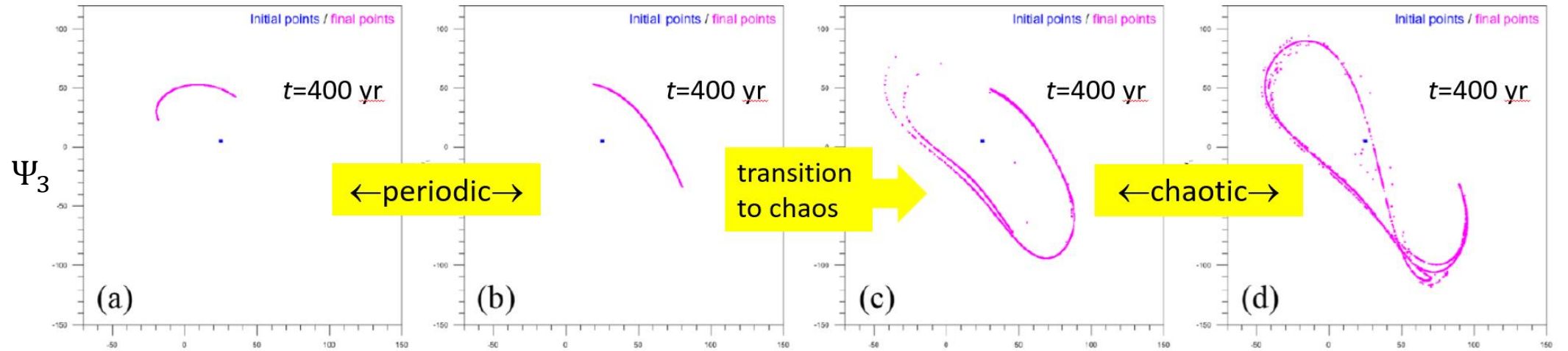
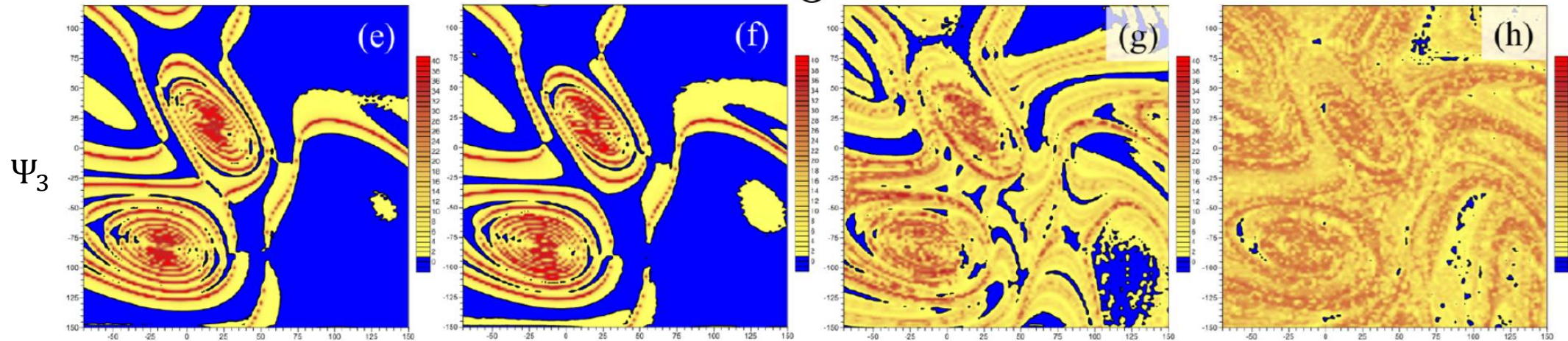
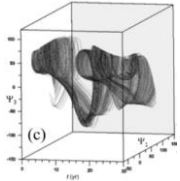
$\gamma=1.1, T_p=30$ yr,

$\varepsilon=0.05$ ●

this is the *periodically forced chaotic* case shown in a previous slide

$\varepsilon=0.1$ ●

$\varepsilon=0.2$ ●



Note that here $\sigma > 1$ for all the initial points contained in the small hypercube

Note that the PBAs of a periodically forced dissipative system are always periodic, but the system can be either chaotic or nonchaotic depending on its parameter values

Periodic forcing

Pierini, Chekroun & Ghil (2018, *Nonlin. Processes Geophys.*)

How to recognize chaos?

Pierini, Chekroun & Ghil (2018) proposed a cross-correlation diagnostics to investigate the transition to chaos

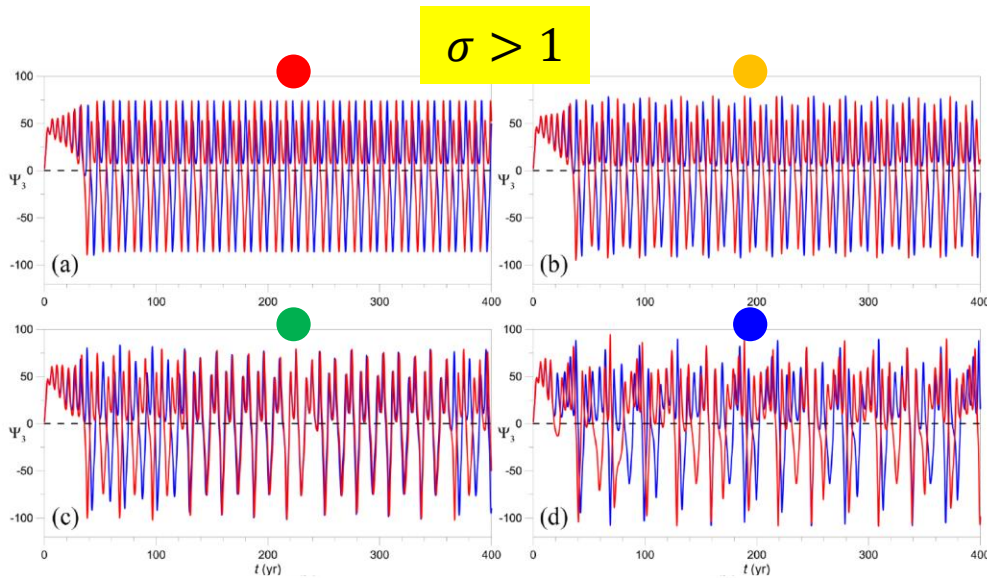


Figure 9. Increasing instability of trajectories as ε increases. Time evolution of $\Psi_3(t)$ for the trajectory initialized at the point P_2 (red line) and at a nearby point (blue line) for $\gamma = 1.1$, $T_p = 30$ years, and (a–d) $\varepsilon = 0, 0.05, 0.1$ and 0.2 .

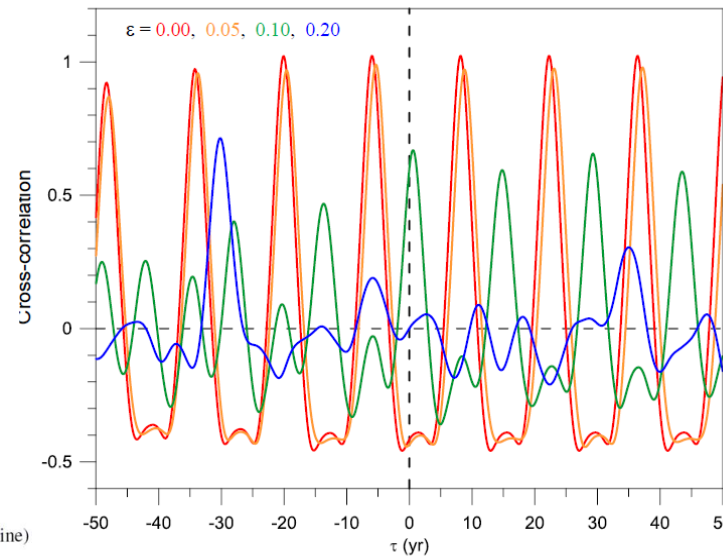


Figure 10. Cross-correlation $c(\tau)$ between the two initially nearby trajectories shown in Fig. 9a–d, computed for the centered and normalized anomalies $\tilde{\Psi}_3$, according to Eq. (7), for $\varepsilon = 0$ (red line), 0.05 (orange line), 0.1 (green line) and 0.2 (blue line).

$$\tilde{\Psi}_3 = \frac{\Psi_3 - \bar{\Psi}_3}{\zeta_3},$$

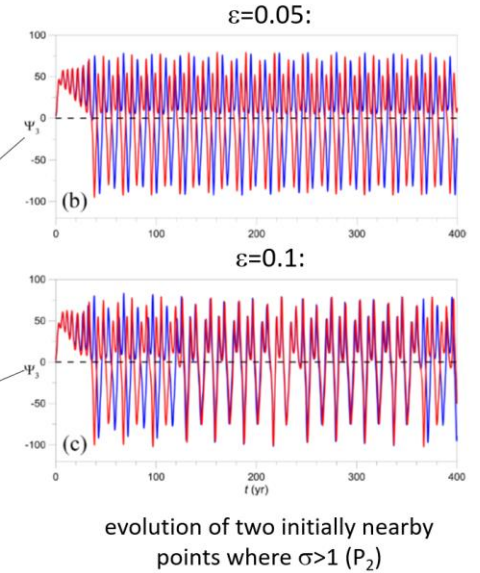
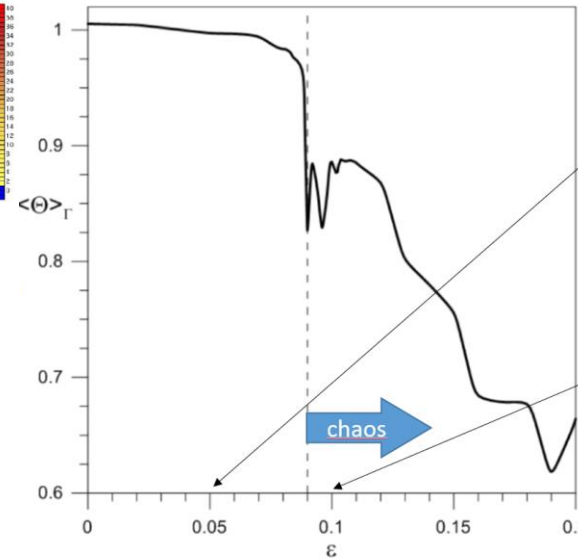
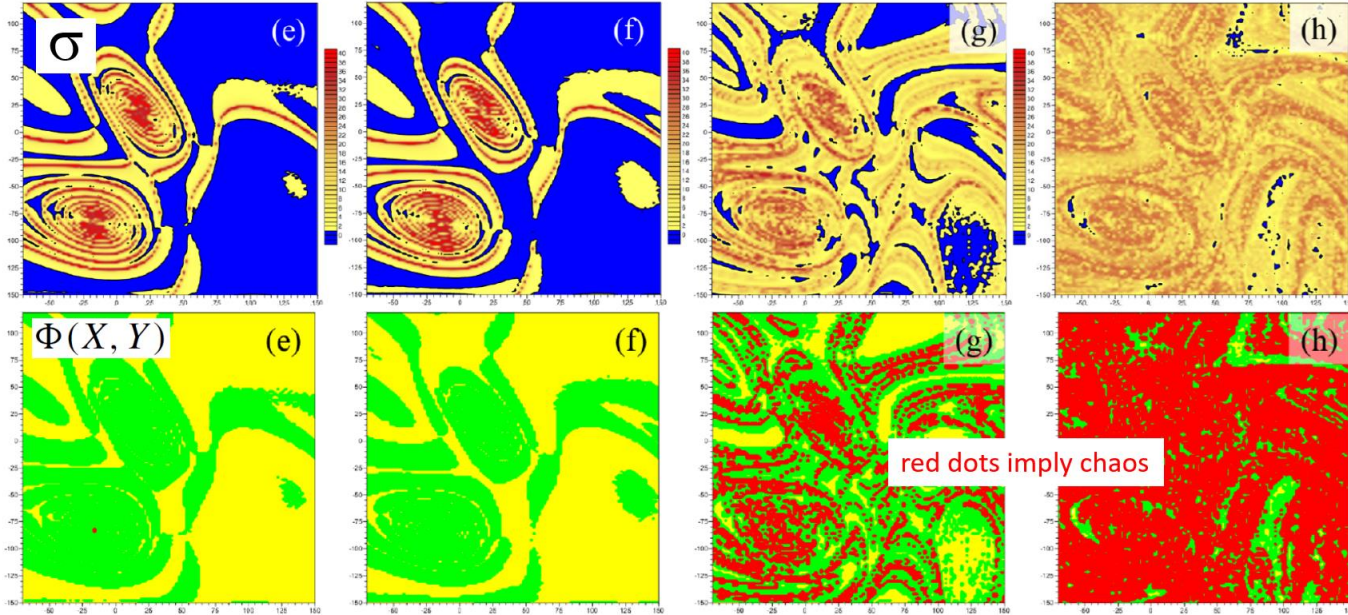
$$c(X, Y, \tau) = \frac{1}{T_* - 2T} \int_T^{T_* - T} \tilde{\Psi}_3(X, Y, t) \tilde{\Psi}_3(X', Y', t + \tau) dt$$

$$\Theta(X, Y) = \max \{c(X, Y, \tau) : \tau \in [-T, T]\}$$

$$\begin{cases} \text{if } \sigma \leq 1 & \Theta \cong 1 \\ \text{if } \sigma > 1 & \begin{cases} \Theta \cong 1 & \text{if the PBA is non-chaotic} \\ \Theta \ll 1 & \text{if the PBA is chaotic} \end{cases} \end{cases}$$

Periodic forcing

Pierini, Chekroun & Ghil (2018, *Nonlin. Processes Geophys.*)



$$\Phi(X, Y) = \begin{cases} 1 & \text{if } \sigma \leq 1 \text{ (yellow), } \Theta_0 = 0.8 \\ 2 & \text{if } \sigma > 1 \text{ and } \Theta > \Theta_0 \text{ (green),} \\ 3 & \text{if } \sigma > 1 \text{ and } \Theta \leq \Theta_0 \text{ (red).} \end{cases}$$

↑
chaos

This cross-correlation diagnostics is a useful tool **also** for autonomous and aperiodically forced systems

The problem

The model

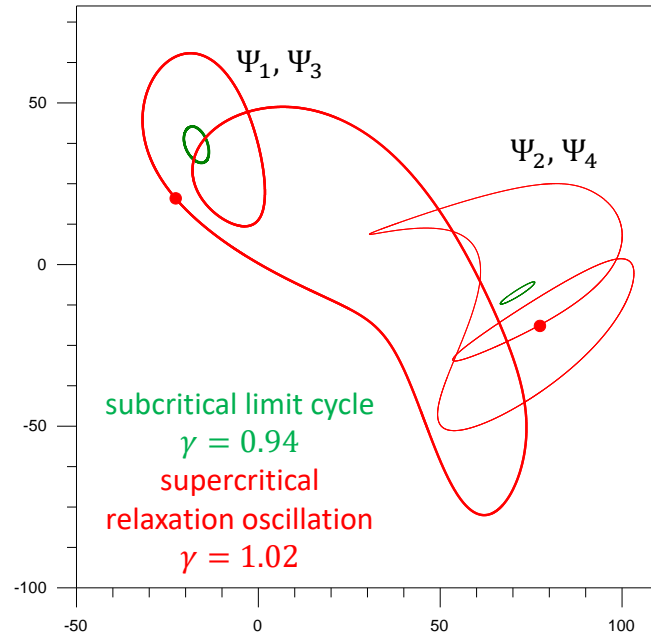
- Periodic forcing
- **Aperiodic forcing**
- Monotonically drifting forcing

Conclusions

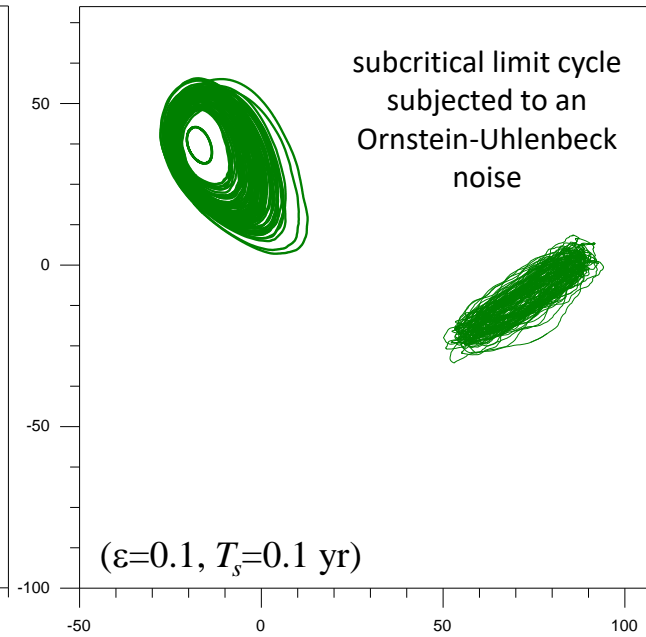
$$G(t) = \gamma \left[1 + \varepsilon \frac{\zeta(t)}{\sigma_\zeta} \right]$$

Aperiodic forcing

Pierini (2012, *Phys. Rev. E*)

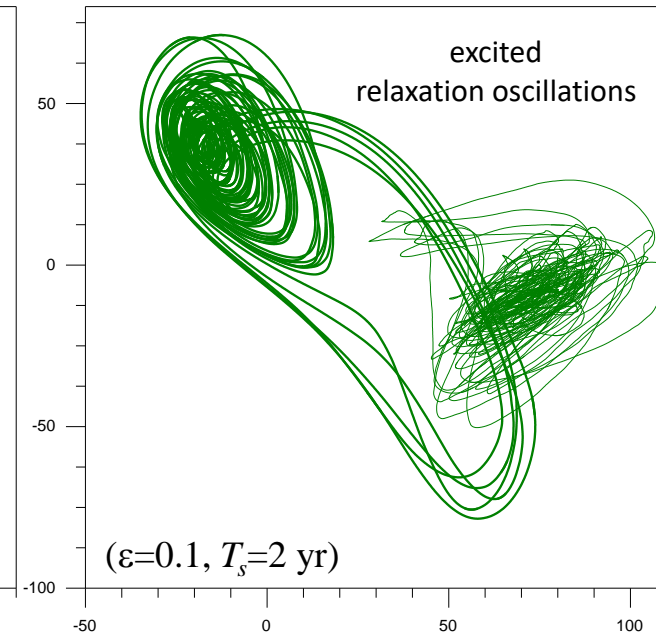


autonomous system



perturbed excitable* system,
no relaxation oscillations

*subcritical



perturbed excitable system,
excited relaxation oscillations

Coherence Resonance

(Pikovsky and Kurths, 1997)

Aperiodic forcing

Pierini, Ghil & Chekroun
(2016, *J. Climate*)

$$G(t) = \gamma[1 + \varepsilon f(t)]$$

This idealized aperiodic forcing mimics the **North Pacific multidecadal variability** in view of a minimal representation of the **Kuroshio Extension low-frequency variability**

The **coherence resonance mechanism** cast in a deterministic framework is clearly playing a relevant role in shaping the PBAs

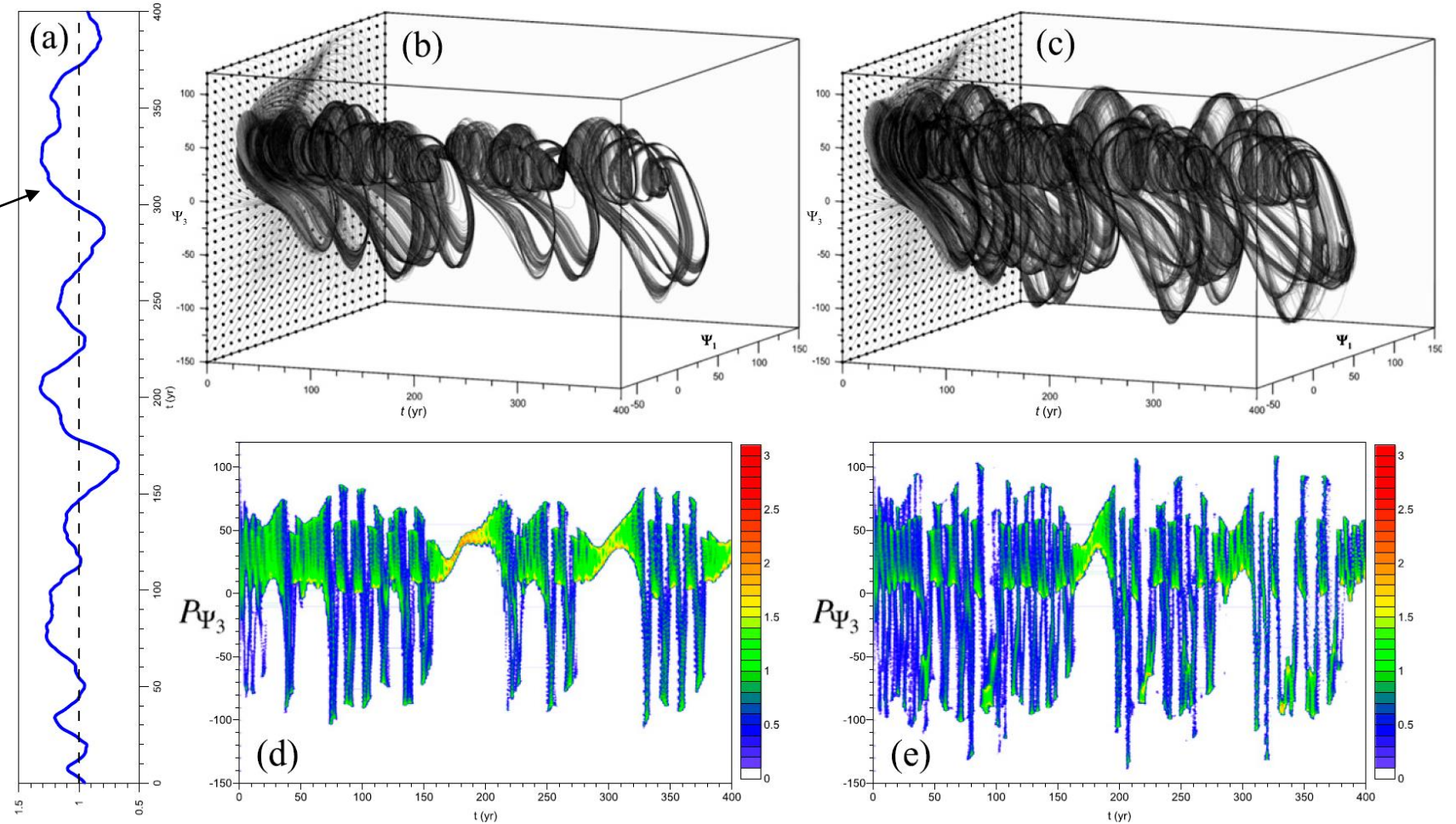
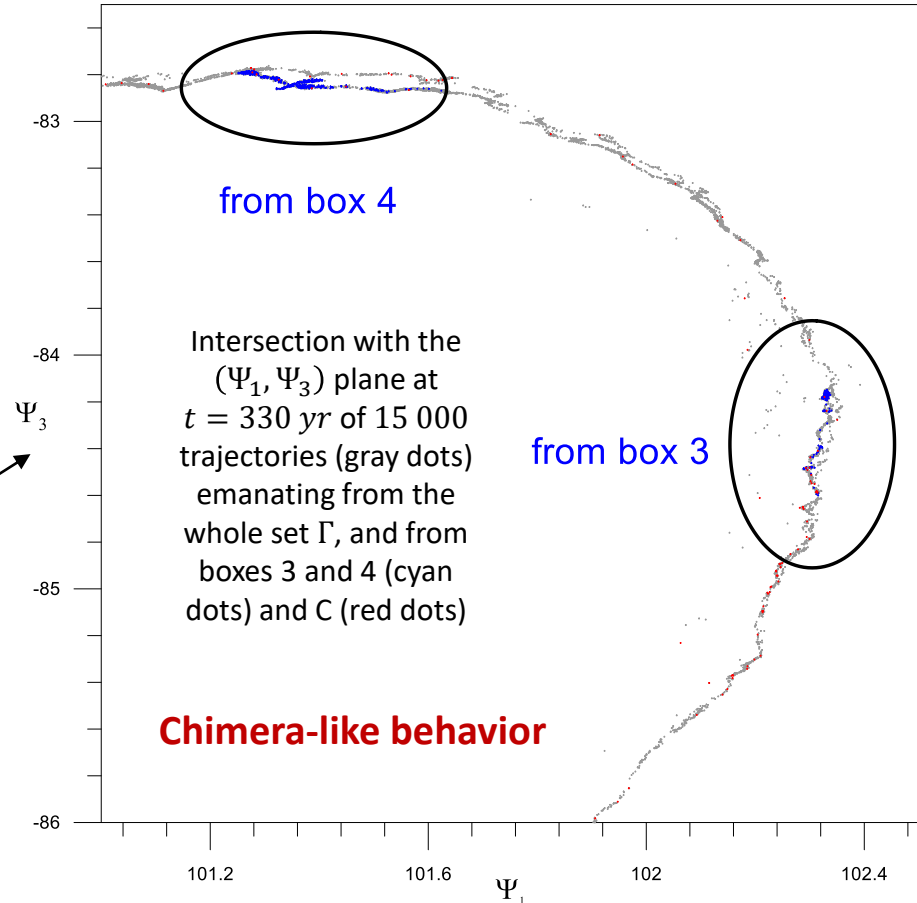
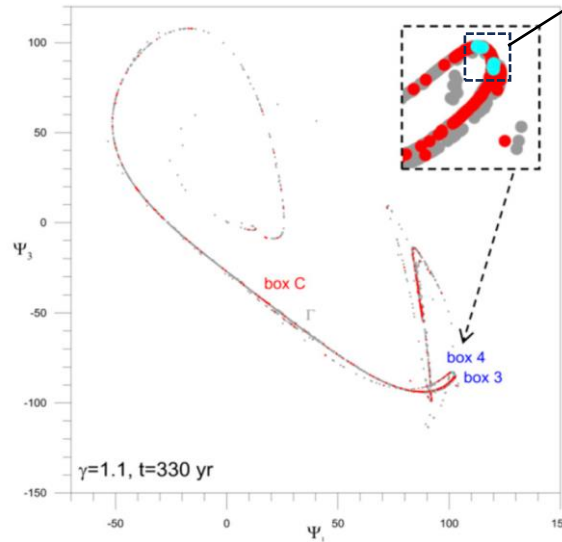
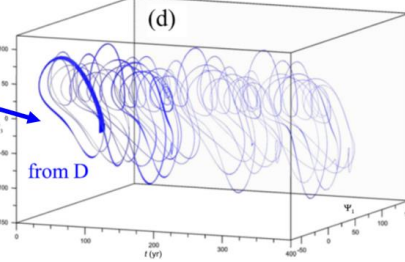
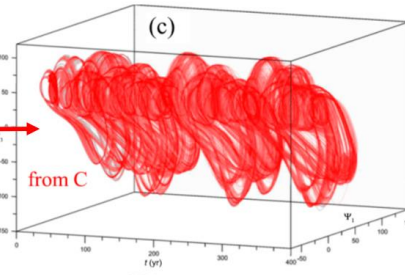
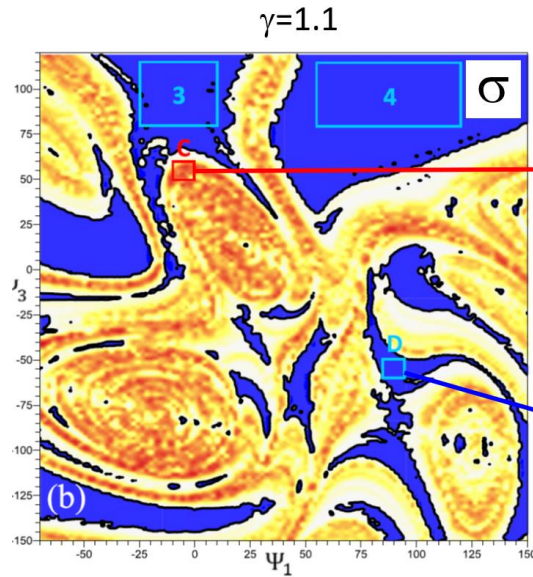


FIG. 2. Ensemble behavior of forced solutions of the double-gyre ocean model. (a) Time dependence of the total forcing $1 + \varepsilon f(t)$, for $\varepsilon = 0.2$. (b),(c) Evolution of 644 initial states emanating from the subset Γ in the (Ψ_1, Ψ_3) plane for (b) $\gamma = 0.96$ and (c) $\gamma = 1.1$. (d),(e) Corresponding time series of P_{Ψ_3} . The set Γ is given by $\{-70 \leq \Psi_1 \leq 150\} \times \{-150 \leq \Psi_3 \leq 120\}$.

Aperiodic forcing

Pierini, Ghil & Chekroun
(2016, *J. Climate*)



restricted regions **over the PBA** exist in which

two nearby points **diverge** during their evolution and

two others **do not**

The corresponding subsets are intertwined over the PBA

Aperiodic forcing

This is another example
of idealized hindcast

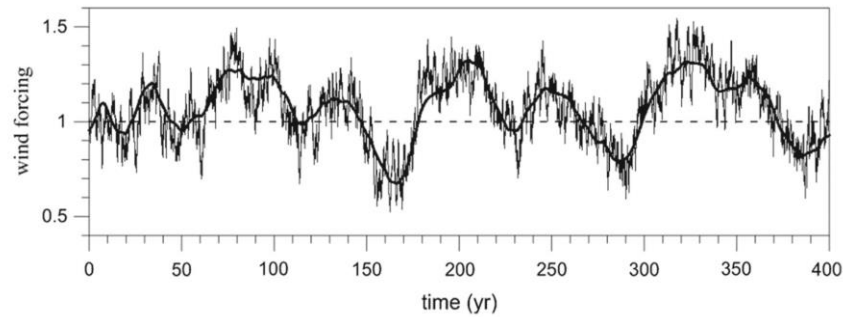


Fig. 1 Thick line: $(1 + H_{LF})$ defined in (2, 3) representing a schematic North Pacific multidecadal variability. Thin line: total time dependence $G/\gamma = H$

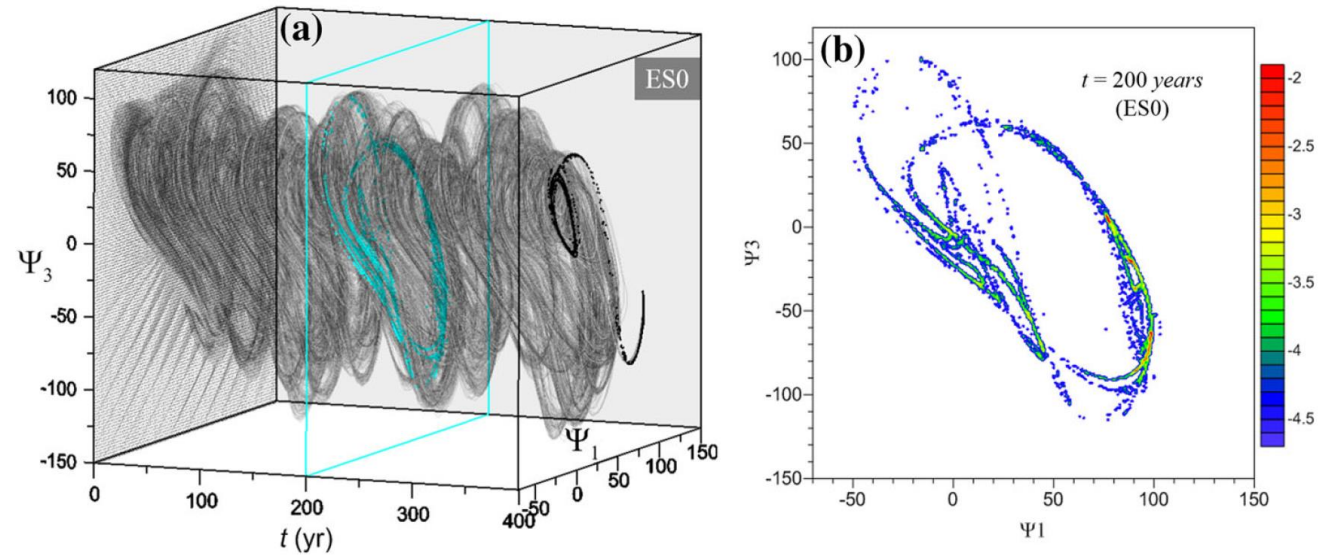
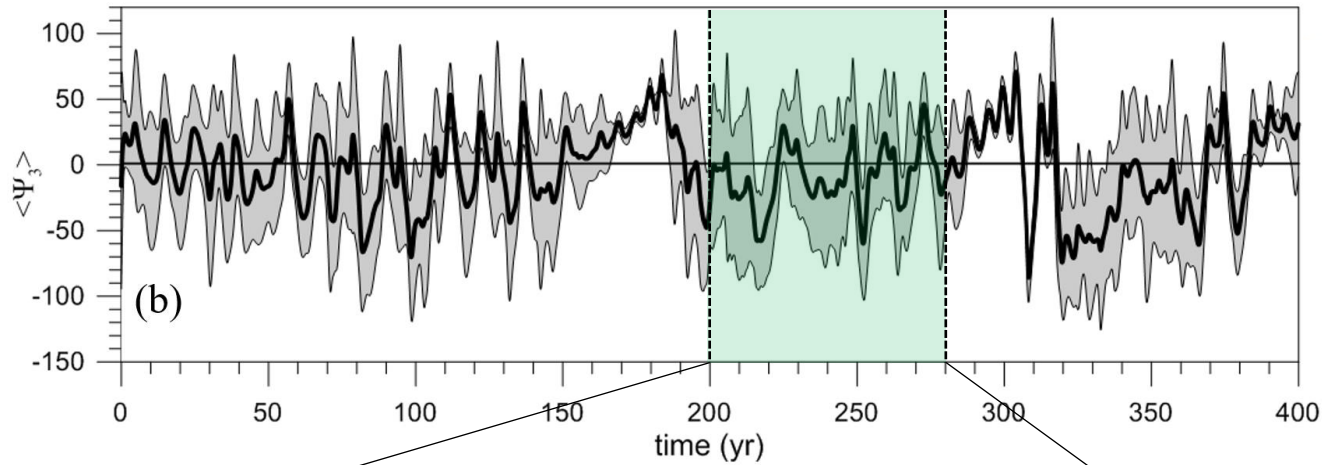


Fig. 3 **a** Grey lines: graphical representation (750 trajectories) of the PBA of system (1) subjected to the forcing $\gamma H \mathbf{w}$; in the same panel the intersection of the PBA –as determined by 15,000 trajectories– with Γ is shown at $t = 0$ (black dots), $t = 200$ years (cyan dots) and $t = 400$ years (black dots). **b** Map of the decimal logarithm of the probability of localization of the PBA trajectories, P_i in Γ at $t = 200$ years (white colour corresponds to minus infinity)

Aperiodic forcing

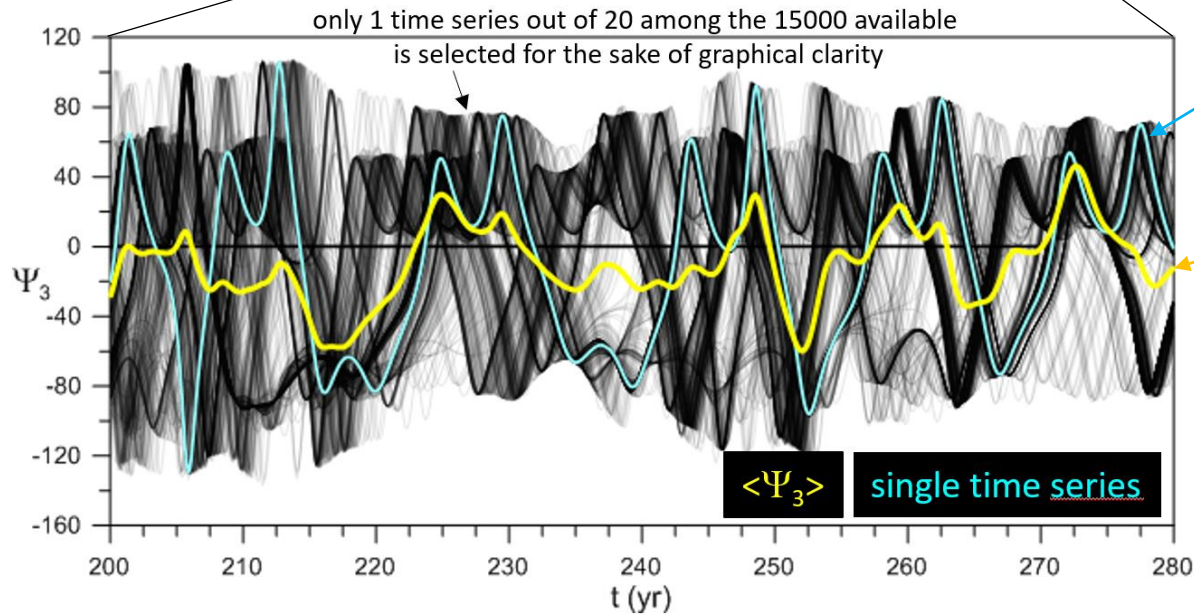
Pierini (2020, *J. Stat. Phys.*)



... disentangling the intrinsic and forced variability ...

“intrinsic variability” (ensemble spread σ)

■ “forced variability” (ensemble mean $\langle \Psi_3 \rangle$)



A single Ψ_3 -time series evidences an **intrinsic relaxation oscillation** paced and **modulated by the forcing**

The PBA shows that **the phases of the relaxation oscillations are not distributed randomly** but are, instead, **clustered in some groups** due to the pacing of the time-dependent forcing

Thus, the ensemble mean feels strongly the clustering of the ROs

→ The ensemble mean depends crucially on intrinsic aspects of the LF variability, therefore, it cannot be unambiguously identified with the FV if internal modes of variability are present (as happens very often)

The ensemble mean of a high-resolution global OGCM is expected to depend less dramatically on the intrinsic dynamics, e.g., as in the **OCCIPUT** ensemble simulation (Penduff et al. 2014, 2019, etc.). But even in that case the ensemble mean is likely to preserve some imprint of the intrinsic dynamics in specific geographical regions

The problem

The model

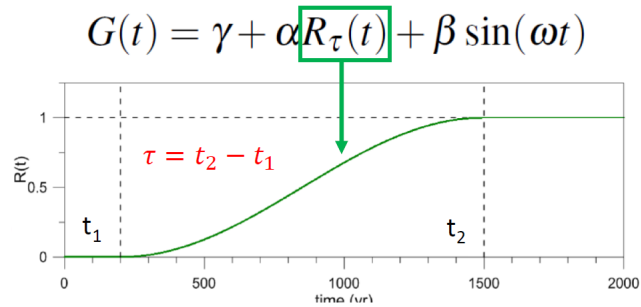
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Conclusions

Monotonically drifting forcing

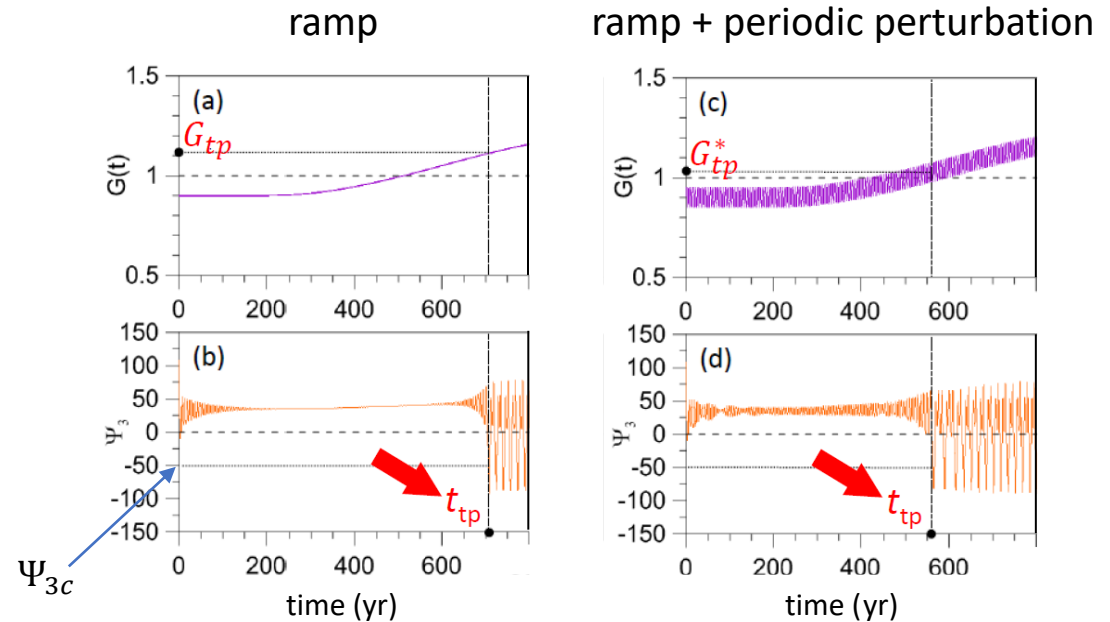
Definition of the time-dependent forcing

- Ramp $R(t)$ given by half-a-cosine function
- Tipping point time t_{tp}
- Corresponding forcing amplitude G_{tp} ($\beta = 0$) or G_{tp}^* ($\beta \neq 0$)



For each value of τ , an ensemble simulation (ES) of 168 members differing only by their initial points is computed to estimate the model's PBA

The **monotonically increasing component** stands for the effect of amplification in the midlatitude winds due to anthropogenic warming, while the **small periodic perturbations** can be thought of as the seasonal-to-interannual variability in the wind stress



The definition of a TP for the time-dependent forcing is extended to an ES as follows: t_{tp} is the time at which, for the first time in any of the ES members, $\Psi_3 < \Psi_{3c}$

Monotonically drifting forcing

Impact of ramp steepness δ

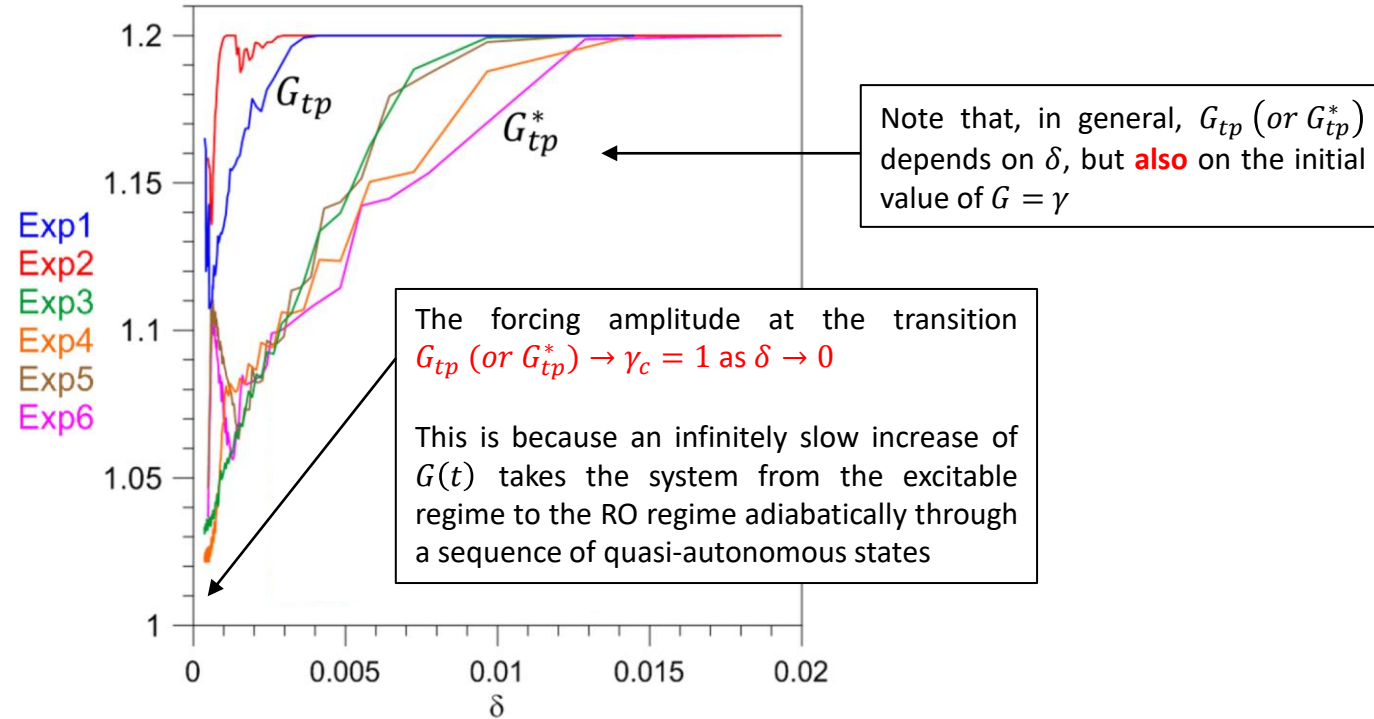
The ramp steepness δ of the forcing $G(t)$ can also be thought of as the drift rate:

$$\delta = \alpha R'_\tau \Big|_{t=(t_1+t_2)/2}$$

- Forcing level G_{tp} at the TPs **decreases** with ramp length τ and **increases** overall with ramp steepness δ .
- A periodic perturbation lowers the forcing level required to reach a TP: $G_{tp}^* \leq G_{tp}$.
- **Surprises occur where the PBA splits.**

In all experiments the final value of G is 1.2

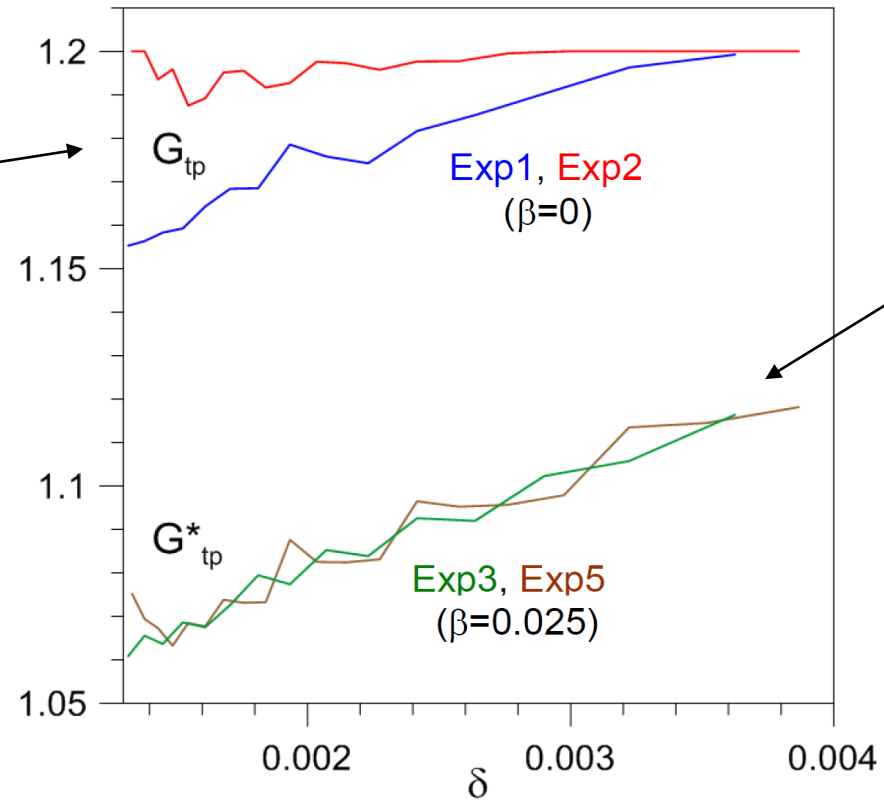
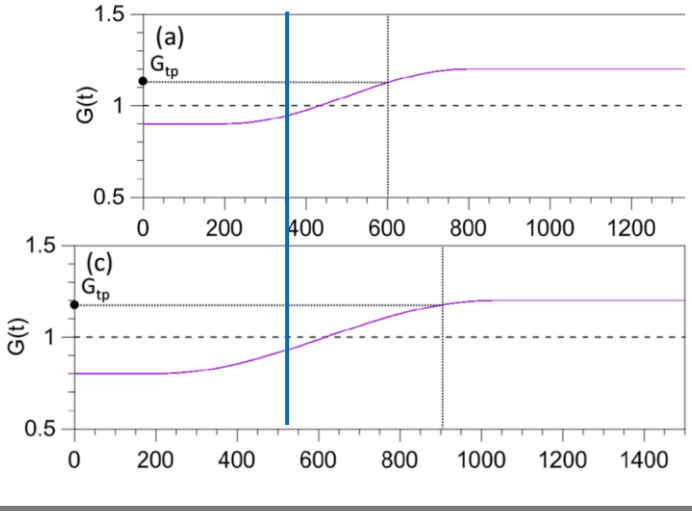
Numerical experiment	γ	α	β	T (yr)
Exp1	0.9	0.3	0	—
Exp2	0.8	0.4	"	—
Exp3	0.9	0.3	0.025	5
Exp4	"	"	0.050	"
Exp5	0.8	0.4	0.025	"
Exp6	"	"	0.050	"



Monotonically drifting forcing

G_{tp} (or G_{tp}^*) depends on δ
but **also** on the initial value of $G = \gamma$

	γ	α
Exp1	0.9	0.3
Exp2	0.8	0.4



However, if the forcing is perturbed by a periodic component ($\beta \neq 0$) the scenario changes drastically:

- The forcing level at the tipping point decreases substantially
- over a certain range of δ -values, G_{tp} (or G_{tp}^*) depends *only* on the drift rate δ !

rate-induced tipping in an excitable system

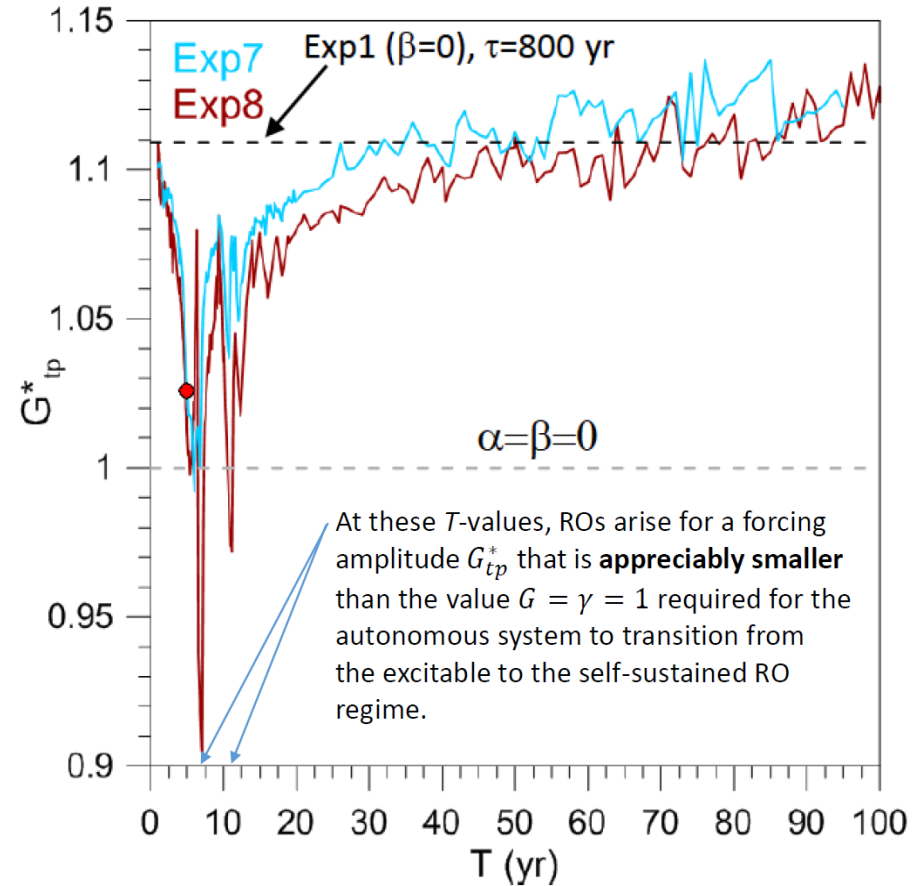
e.g., see Ashwin et al. (2012), Vanselow, Wieczorek, and Feudel (2019), etc.

Monotonically drifting forcing

Nonlinear resonance

Exp7 & Exp8 differ from EXP1 in that a periodic forcing is now present, with T that varies from 1 to 100 yr, while $\tau = 800$ yr is constant.

The abrupt reduction of G_{tp}^* occurs for periods T that are comparable to the typical time scale of the ROs; thus, **nonlinearly resonant**-like behavior seems to occur.



Clustering of Trajectories and Phase Coherence

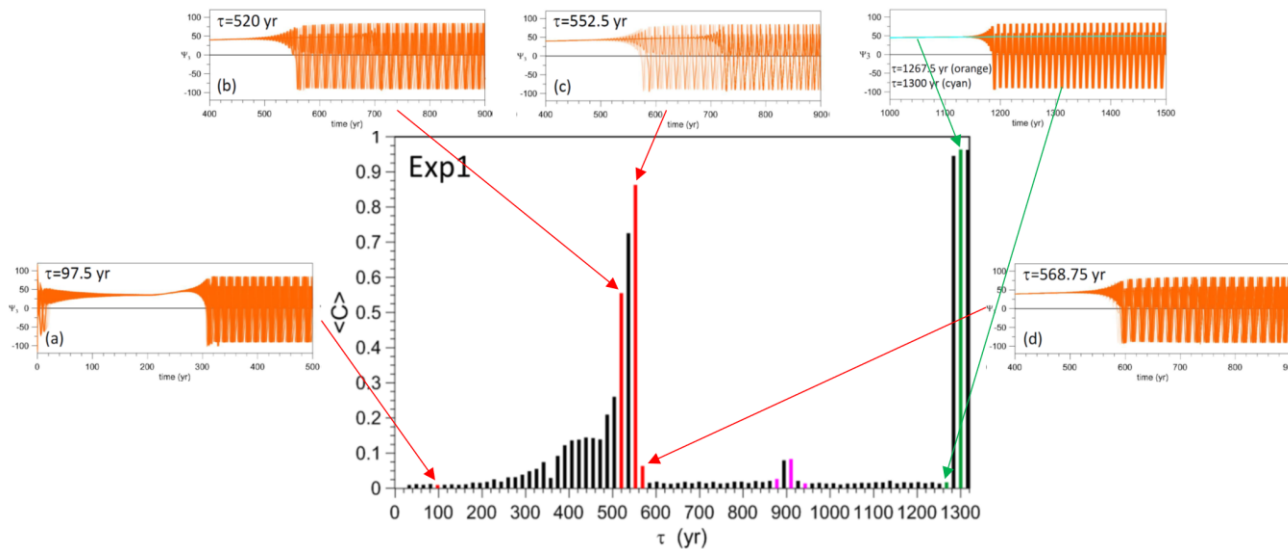
To study these aspects of the model's ROs, Pierini (2014) proposed to use the **parameter** \bar{C} , given by

$$\bar{C}(r, \tau) = \frac{1}{T_0} \int_{t_{tp}(\tau)}^{t_{tp}(\tau)+T_0} C(r, t, \tau) dt \quad \text{with} \quad C(r, t, \tau) = \frac{1}{N^2} \sum_{i,j} H \left[r - \left| \Psi_{\tau}^{(i)}(t) - \Psi_{\tau}^{(j)}(t) \right| \right]$$

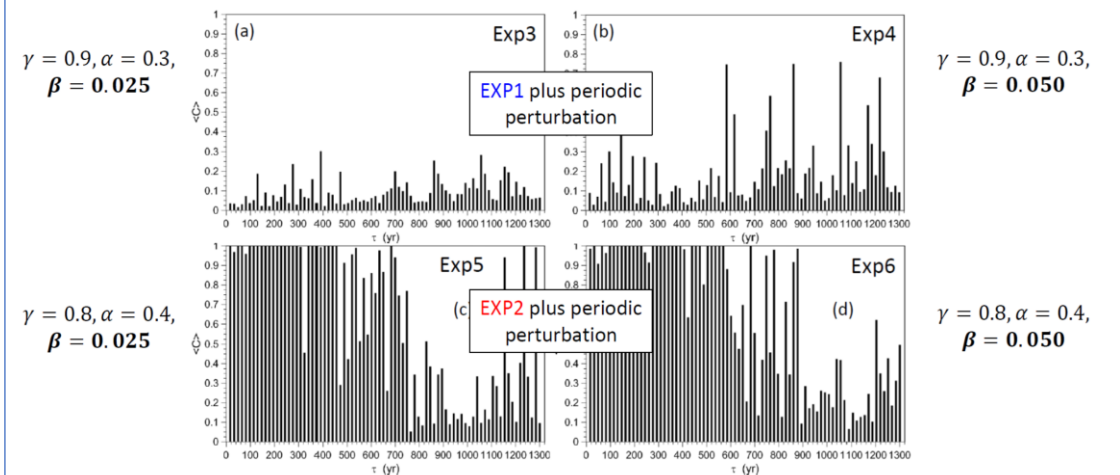
to obtain information about the **clustering** and **phase dependence** of ensemble members. Here $H(r - x)$ is a Heaviside function that counts proximity of orbit pairs.

$(1/n) \leq \bar{C} \leq 1$, where n is the number of clusters defined by the distance r .
For $\bar{C} = 1$ one has only one cluster and, therefore, **TIID**.

TIID: *total independence from the initial data*



Effect of the periodic perturbation on the clustering



The degree of clustering is basically preserved under periodic perturbation.
Thus, TIID appears to be a fairly robust feature

Conclusions

The pullback attractor is a mathematical tool required to characterize a nonautonomous dissipative dynamical system. Thus, it is also an **extremely power tool to reveal fundamental and often unexpected dynamical features**

The analysis of the PBAs of a low-order ocean model subjected to

- **periodic forcing** has revealed:

- the cyclostationarity and cycloergodicity of chaotic solutions
- cases of generalized synchronization
- the ways in which chaos is induced by the forcing

- **aperiodic forcing** has revealed:

- the role played by coherence resonance in shaping the response
- the difficulty of unambiguously disentangling the forced and intrinsic variability in the presence of internal modes of variability

- **monotonically drifting forcing** has revealed:

- the peculiar way in which the tipping points, leading to a supercritical response, occur
- cases of rate-induced tipping
- cases of nonlinear resonance
- the clustering of trajectories and phase coherence

In all cases, the **coexistence of completely different evolutions within the same PBA** is found

Future studies include:

- (i) Further analyses on aperiodic and monotonically drifting forcing cases (focusing on rate-induced tipping)
- (ii) Application to phase space topology, templex (**Gisela Charó's talk** on Friday)
- (iii) Application of the same methodologies to the **abrupt late Pleistocene glacial terminations** according to the **deterministic excitation** paradigm (Pierini 2023, *Chaos*)

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Elegantly Modeling Earth’s Abrupt Glacial Transitions

MARCH 7, 2023

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Simple and intuitive model illustrates how climate cycles are influenced by our planet’s orbit.

From the Journal: [Chaos](#)

WASHINGTON, March 7, 2023 – Proxy data – indirect records of the Earth’s climate found in unlikely places like coral, pollen, trees, and sediments – show interesting oscillations approximately every 100,000 years starting about 1 million years ago. Strong changes in global ice volume, sea level, carbon dioxide concentration, and surface temperature indicate cycles of a long, slow transition to a glacial period and an abrupt switch to a warm and short interglacial period.

Milutin Milankovitch hypothesized that the timing of these cycles was controlled by the orbital parameters of the Earth, including the shape of its path around the sun and the tilt of the planet. A slightly closer orbit or more tilted planet could create a small increase in solar radiation and a feedback loop that leads to massive changes in climate. This idea suggests that there may be some predictability in the climate, a notoriously complex system.

In Chaos, by AIP Publishing, Stefano Pierini of Parthenope University of Naples proposed a new paradigm to simplify the verification of the Milankovitch hypothesis.

“The main motivation behind this study was the wish to characterize and illustrate the Milankovitch hypothesis in a simple, elegant, and intuitive way,” Pierini said.

Many models suggest that Milankovitch is correct; however, such methods are often detailed and study specific. They incorporate climate feedback loops – for example, increased ice cover reflects more radiation back into space, leading to further cooling and more ice cover – as threshold crossing rules. This means that an abrupt jump in climate only occurs once a parameter reaches a given tipping point.

Pierini’s “deterministic excitation paradigm” combines the physics concepts of relaxation oscillation and excitability to link Earth’s orbital parameters and the glacial cycles in a more generic way. The relaxation oscillation component describes how the climate slowly returns to its original glacier state after it is disturbed. At that point, the excitability piece of the model captures the external orbital changes and triggers the next glacial cycle.

By using his own threshold crossing rules and adopting a classical energy-balance model, Pierini obtained correct and robust timing of the most recent glacial cycles.

“The application of the deterministic excitation paradigm in the present basic formulation can explain the timing of the last four glacial terminations,” he said. “Extending the same analysis to the whole Pleistocene will be the subject of a future investigation.”

Pierini believes similar methods could be used in other fields of nonlinear science and in connection with other climate phenomena.

###



The Laurentide Ice Sheet covered most of northern North America during glacial periods. Credit: NOAA Great Lakes Environmental Research Laboratory, jan.ucc.nau.edu/~rcb7/nam.html. Permission to re-use. https://www.flickr.com/photos/noaa_glerl/8740576431

Article Title

[The deterministic excitation paradigm and the late Pleistocene glacial terminations](#)

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